Isomorphic Classification of the Spaces of Whitney Functions

ALEXANDER P. GONCHAROV &
MEFHARET KOCATEPE

I. Introduction

Let $K \subset \mathbb{R}$ be a compact set such that $K = \overline{\text{int}\, K}$. By $\mathcal{E}(K)$ we denote the space of infinitely differentiable Whitney functions on K. This is the space of functions $f: K \to \mathbb{R}$ extendable to C^∞-functions on $\mathbb{R}$ equipped with the topology defined by the sequence of norms

$$\|f\|_q = |f|_q + \sup\{\, |(R_y^q f)^{(i)}(x)| \cdot |x-y|^{i-q} : \\ x, y \in K,\ x \neq y,\ i \leq q\,\},\ q = 0, 1, \dots,$$

where $|f|_q = \sup\{\, |f^{(j)}(x)| : x \in K,\ j \leq q\,\}$ and

$$R_y^q f(x) = f(x) - T_y^q f(x) = f(x) - \sum_{k=0}^{q} \frac{f^{(k)}(y)}{k!}(x-y)^k$$

is the Taylor remainder. With

$$U_q = \{\, f \in \mathcal{E}(K) : \|f\|_q \leq 1\,\},$$

the sequence (U_q) need not decrease, but the sets εU_q with $\varepsilon > 0$ and $q \in \mathbb{N}$ constitute a basis of neighborhoods of zero in $\mathcal{E}(K)$. It was shown in [20] by Tidten and in [25] by Vogt that the space $\mathcal{E}(K)$ is isomorphic to the space

$$s = \left\{ x = (\xi_n) : \|x\|_q = \sum_{n=1}^{\infty} |\xi_n| n^q < \infty\ \forall q \right\}$$

of rapidly decreasing sequences if and only if there is a continuous extension operator $L: \mathcal{E}(K) \to C^\infty(\mathbb{R})$.

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. We consider compact sets of the following type. For two sequences (a_n), (b_n) such that $0 < \cdots < b_{n+1} < a_n < b_n < \cdots < b_1 < 1$, let $I_n = [a_n, b_n]$ and $K = \{0\} \cup \bigcup_{n=1}^{\infty} I_n$. By ψ_n we denote the length of I_n; $h_n = a_n - b_{n+1}$ is the distance between I_n and I_{n+1}. In what follows we restrict ourselves to the case

$$\psi_n \searrow 0, \quad h_n \searrow 0, \quad \psi_n \leq h_n, \qquad n \in \mathbb{N}, \tag{1}$$

$$\exists Q \in \mathbb{N} : h_n \geq b_{n+1}^Q, \qquad n \in \mathbb{N}. \tag{2}$$

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An equivalent form of (2) is

$$\exists Q \in \mathbb{N} : h_n \geq b_n^Q, \quad n \in \mathbb{N}. \tag{3}$$

In fact, (3) trivially implies (2). On the other hand, if (2) holds, then

$$b_n = b_{n+1} + h_n + \psi_n \leq h_n^{1/Q} + 2h_n \leq 3h_n^{1/Q},$$

which implies (3).

We list here some identities about Taylor polynomials and remainders that will be used in this paper. The proofs of these identities can be found, for example, in [15]:

$$(R_y^q f)^{(i)}(x) = R_y^{q-i} f^{(i)}(x) = f^{(i)}(x) - \sum_{j=i}^{q} \frac{f^{(j)}(y)}{(j-i)!}(x-y)^{j-i}; \tag{4}$$

$$R_y^q R_z^q f(x) = R_y^q f(x); \tag{5}$$

$$T_y^q f(x) - T_a^q f(x) = T_y^q (R_a^q f)(x). \tag{6}$$

If $f \in C^{q+1}[a,b]$ and $x, y \in [a,b]$, then for some $\xi, \eta \in [a,b]$ we have

$$(R_y^q f)^{(i)}(x) = (f^{(q)}(\xi) - f^{(q)}(y))\frac{(x-y)^{q-i}}{(q-i)!} = f^{(q+1)}(\eta)\frac{(x-y)^{q-i+1}}{(q-i+1)!}. \tag{7}$$

The next lemma can be derived easily from Lemma 1 in [21]; see also Lemma 1 in [9].

LEMMA 1. *Let I be any closed interval in $\mathbb{R}$ with* length$(I) \geq \delta_0$ *and let $p \leq k \leq r$ be given. Then there exist two constants C_1, C_2 such that*

$$|f^{(k)}(x)| \leq C_1 \delta^{-k+p} |f|_p + C_2 \delta^{r-k} |f|_r \quad \forall f \in C^r(I), \ \forall \delta \in (0, \delta_0], \ \forall x \in I.$$

LEMMA 2. *Let $K \subset \mathbb{R}$ be a compact set containing $r+1$ distinct points $x_0, \ldots, x_r$ such that $x_0 < x_1 < \cdots < x_r$ and $h := x_1 - x_0 \leq x_2 - x_1 \leq \cdots \leq x_r - x_{r-1} =: H$. Let $f \in \mathcal{E}^r(K)$, $1 \leq k \leq r$. Then*

$$|f^{(k)}(x_0)| \leq C_3 h^{-k} |f|_0 + C_4 H^{r-k} \|f\|_r,$$

where C_3 and C_4 depend only on k and r.

Here $\mathcal{E}^r(K)$ is the Banach space of r-times differentiable Whitney functions equipped with the norm $\|\cdot\|_r$.

Proof. We will use the Vandermonde determinant

$$V(a_0, a_1, \ldots, a_n) = \prod_{i<j} (a_j - a_i)$$

and elementary symmetric functions. For $1 \leq k \leq n$,

$$S_k = S_k(a_1, \ldots, a_n) = a_1 a_2 \ldots a_k + \cdots + a_{n-k+1} \ldots a_n$$

is the sum of $\binom{n}{k}$ terms, where each term is the product of k factors without repetition. Then

$$\begin{vmatrix} a_1 & a_1^2 & \cdots & a_1^{k-1} & a_1^{k+1} & \cdots & a_1^n \\ a_2 & a_2^2 & \cdots & a_2^{k-1} & a_2^{k+1} & \cdots & a_2^n \\ \vdots & & & & & & \\ a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{k-1} & a_{n-1}^{k+1} & \cdots & a_{n-1}^n \end{vmatrix}$$
$$= a_1 \ldots a_{n-1} S_{n-k}(a_1, \ldots, a_{n-1}) V(a_1, \ldots, a_{n-1}).$$

To show this, we denote the above determinant by B_k. We consider the expression $a_1 \ldots a_n V(a_1, \ldots, a_n)$ as a polynomial in $x = a_n$:

$$\begin{aligned} a_1 \ldots a_n V(a_1, \ldots, a_n) &= a_1 \ldots a_n \prod_{i<j} (a_j - a_i) \\ &= \underbrace{a_1 \ldots a_{n-1} V(a_1, \ldots, a_{n-1})}_{\alpha} a_n (a_n - a_1) \ldots (a_n - a_{n-1}) \\ &= \alpha a_n (a_n^{n-1} - S_1(a_1, \ldots, a_{n-1}) a_n^{n-2} + S_2 a_n^{n-3} + \cdots \\ &\qquad + (-1)^{n-k} S_{n-k} a_n^{k-1} + \cdots + (-1)^{n-1} S_{n-1}) \\ &= \cdots + (-1)^{n-k} \alpha S_{n-k}(a_1, \ldots, a_{n-1}) a_n^k + \cdots . \end{aligned}$$

We note that neither α nor $S_{n-k}(a_1, \ldots, a_{n-1})$ contains a_n. On the other hand, clearly

$$a_1 \ldots a_n V(a_1, \ldots, a_n) = \begin{vmatrix} a_1 & \ldots & a_1^k & \ldots & a_1^n \\ \vdots & & & & \\ a_n & \ldots & a_n^k & \ldots & a_n^n \end{vmatrix}.$$

When we expand this determinant with respect to the kth column, we see that none of the minors of $a_1^k, \ldots, a_{n-1}^k$ contain the term a_n^k. So this expansion becomes $\cdots + (-1)^{k+n} a_n^k B_k$, where B_k does not contain any a_n. So, in the two different expansions of $a_1 \ldots a_n V(a_1, \ldots, a_n)$, we compare the coefficients of a_n^k and obtain

$$(-1)^{n-k} \alpha S_{n-k} = (-1)^{k+n} B_k,$$

which gives the desired result.

Let us use the notation $\pi_r(x) = (x - x_0)(x - x_1) \cdots (x - x_r)$, $h_i = x_i - x_{i-1}$, $f_i^{(k)} = f^{(k)}(x_i)$, $f_i = f(x_i)$, and $F_i = f_i - f_0 - R_{x_0}^r f(x_i)$ for $i = 0, 1, \ldots, r$. Consider the system of equations

$$f_0^{(1)}(x_i - x_0) + \cdots + \frac{f_0^{(k)}}{k!}(x_i - x_0)^k + \cdots + \frac{f_0^{(r)}}{r!}(x_i - x_0)^r = F_i, \quad i = 1, \ldots, r,$$

in the "unknowns" $f_0^{(k)}/k!$, $k = 1, \ldots, r$. Let $\Delta = V(x_0, x_1, \ldots, x_r)$ and

$$\Delta_k = \begin{vmatrix} x_1 - x_0 & \ldots & (x_1 - x_0)^{k-1} & F_1 & (x_1 - x_0)^{k+1} & \ldots & (x_1 - x_0)^r \\ \vdots & & & & & & \\ x_r - x_0 & \ldots & (x_r - x_0)^{k-1} & F_r & (x_r - x_0)^{k+1} & \ldots & (x_r - x_0)^r \end{vmatrix}.$$

Let M_i denote the minor corresponding to the (i, k)th entry. Then

$$\Delta_k = \sum_{i=1}^{r} (-1)^{i+k} F_i M_i$$

with

$$M_i = (x_1 - x_0)\dots(x_{i-1} - x_0)(x_{i+1} - x_0)\dots(x_r - x_0) S_{r-k}(\cdots) V(\cdots),$$

where $\cdots$ in $S_{r-k}(\cdots)$ and $V(\cdots)$ is $x_1 - x_0, \dots, x_{i-1} - x_0, x_{i+1} - x_0, \dots, x_r - x_0$. Since

$$V(\cdots) = (-1)^{r-i} \frac{V(x_1, \dots, x_r)(x_i - x_0)}{\pi_r'(x_i)},$$

we have

$$M_i = (-1)^{r-i} \frac{V(x_0, \dots, x_r)}{\pi_r'(x_i)} S_{r-k}(\cdots)$$

and, by Cramer's rule,

$$\frac{f_0^{(k)}}{k!} = (-1)^{r+k} \sum_{i=1}^{r} F_i \frac{S_{r-k}(\cdots)}{\pi_r'(x_i)}, \qquad k = 1, 2, \dots, r.$$

Setting $h_j = x_j - x_{j-1}$ for $j = 1, \dots, r$, we have

$$\begin{aligned} |\pi_r'(x_i)| &= |(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_r)| \\ &= (h_1 + \cdots + h_i)(h_2 + \cdots + h_i) \\ &\qquad \dots h_i h_{i+1}(h_{i+1} + h_{i+2})\dots(h_{i+1} + \cdots + h_r) \\ &\geq h_i^i h_{i+1} \dots h_r. \end{aligned}$$

Next we estimate $|S_{r-k}(\cdots)|$. The term in $S_{r-k}(\cdots)$ with maximal absolute value is $(x_r - x_0)\dots(x_{k+1} - x_0)$, and the number of terms in $S_{r-k}(\cdots)$ is $\binom{r-1}{r-k}$. Hence

$$\begin{aligned} |S_{r-k}(\cdots)| &\leq \frac{(r-1)!}{(k-1)!\,(r-k)!} r h_r (r-1) h_{r-1} \dots (k+1) h_{k+1} \\ &= \underbrace{\frac{(r-1)!\,r!}{(k-1)!\,(r-k)!\,k!}}_{C_{r,k}} h_{k+1} \dots h_r, \end{aligned}$$

which implies

$$\left| \frac{S_{r-k}}{\pi_r'(x_i)} \right| \leq C_{r,k} \frac{h_{k+1} h_{k+2} \dots h_r}{h_i^i h_{i+1} \dots h_r}.$$

We also have

$$\begin{aligned} |F_i| &= |f(x_i) - f(x_0) - R_{x_0}^r f(x_i)| \leq 2|f|_0 + \|f\|_r |x_i - x_0|^r \\ &\leq 2|f|_0 + \|f\|_r (h_1 + \cdots + h_r)^r \leq 2|f|_0 + \|f\|_r i^r h_i^r. \end{aligned}$$

Thus

$$\left| \frac{F_i S_{r-k}}{\pi_r'} \right| \leq 2|f|_0 C_{r,k} \frac{h_{k+1} h_{k+2} \dots h_r}{h_i^i h_{i+1} \dots h_r} + \|f\|_r i^r \frac{h_i^r h_{k+1} \dots h_r}{h_i^i h_{i+1} \dots h_r} C_{r,k}.$$

Now

$$\frac{h_{k+1}h_{k+2}\dots h_r}{h_i^i h_{i+1}\dots h_r} \le \frac{1}{h_1 h_2 \dots h_k} \le h_1^{-k},$$

since $h_1 \dots h_k h_{k+1} \dots h_r \le h_i^i h_{i+1} \dots h_r$ and

$$\frac{h_i^r h_{k+1}\dots h_r}{h_i^i h_{i+1}\dots h_r} \le \begin{cases} \frac{h_i^{r-i}}{h_{i+1}\dots h_k} \le \frac{h_i^{r-i}}{h_i^{k-i}} = h_i^{r-k} \le h_r^{r-k} & \text{if } i < k, \\ h_i^{r-i} h_{k+1}\dots h_i \le h_i^{r-i} h_i^{i-k} = h_i^{r-k} \le h_r^{r-k} & \text{if } i \ge k. \end{cases}$$

Thus

$$|f^{(k)}(x_0)| \le C_3 |f|_0 h_1^{-k} + C_4 \|f\|_r h_r^{r-k},$$

where

$$C_3 = k!\,2rC_{r,k} \quad \text{and} \quad C_4 = k!\,C_{r,k}\sum_{i=1}^{r} i^r. \qquad \square$$

Now Lemma 1 (the case $p = 0$) can be deduced from Lemma 2. In fact, one can take $r + 1$ equidistant points on I with the step $h = \delta/r$ and use equivalence of the norms $|\cdot|_r$ and $\|\cdot\|_r$ on the interval.

LEMMA 3. *Given positive integers N, p, k such that $k \le pN$, there is a constant $C(N, p, k)$ with the following properties: For any closed interval $I \subset \mathbb{R}$ with* length$(I) = \delta_0$ *and for any set of points $a_1, \dots, a_N \in I$, let $G(x) = \prod_{s=1}^{N}(x - a_s)^p$. Then*

$$|G^{(k)}(x)| \le C(N, p, k)\delta_0^{pN-k} \quad \forall x \in I.$$

Proof. Let

$$b_{jp+1} = b_{jp+2} = \dots = b_{jp+p} = a_{j+1}, \quad j = 0, 1, \dots N - 1.$$

Then

$$G(x) = \prod_{i=1}^{pN}(x - b_i)$$

and, by induction,

$$G^{(k)}(x) = \sum_{|A|=Np-k} C(A)\prod_{s\in A}(x - b_s),$$

where A is a subset of $\{1, 2, \dots, Np\}$ and $|A|$ is the number of elements of A. This proves the lemma. $\square$

II. Spaces $\mathcal{E}(K)$ and the Property D_φ

Now we consider a linear topological invariant introduced by Vogt [26] and Tidten [22] (and called DN_φ by them) and by Goncharov and Zahariuta [7; 36; 11].

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function, $\varphi(t) \ge t$. A Fréchet space X with a fundamental increasing system of seminorms $(\|\cdot\|_p)_{p=0}^\infty$ has the property D_φ if $\exists p \in \mathbb{Z}_+$, $\forall q \in \mathbb{N}$, $\exists r \in \mathbb{N}$, $m > 0$, $C > 0$ such that

$$\|f\|_q \le \varphi^m(t)\|f\|_p + \frac{C}{t}\|f\|_r, \quad t > 0, \quad f \in X.$$

Examples of continua of families of pairwise nonisomorphic spaces C^∞ [22; 11] and Whitney functions [12] were found by means of these invariants.

The invariant D_φ appeared as a generalization of the class D_1 (see [31]) or the property DN [23]. In the case $\varphi(t) = t$, D_φ coincides with Ω_2 [35] or $\underline{DN}$ [24].

For each n, we define $J_n = \min\{ j : b_{n+j} \le \psi_n \}$ and assume that either (J_n) is bounded or $J_n \to \infty$ as $n \to \infty$.

THEOREM 1. *Let $J_n \le J$ for each n. Then $\mathcal{E}(K)$ has property D_φ if and only if*

$$\exists M,\ \forall n, \quad \psi_n \ge \varphi^{-M}(h_{n-1}^{-M}). \tag{8}$$

Proof. We suppose that J and M are natural numbers.

Necessity. We have p from D_φ. We let $\mu = Q(p+1)$ and $q = (J+1)\mu$ and find r, m, C according to D_φ. We fix n and define

$$f = f_n = \begin{cases} \left(x \prod_{s=n}^{n+J-1}(x - a_s)\right)^\mu, & x \le b_n, \\ 0, & x \ge a_{n-1}. \end{cases}$$

Since $b_{n+J_n} \le \psi_n$, we have $b_{n+J} \le \psi_n$ for all n.

Because f is a polynomial of degree q on $[0, b_n]$, we trivially have $\|f\|_q \ge |f|_q \ge |f^{(q)}|_0 = q!$. Next we find upper bounds for $\|f\|_p$ and $\|f\|_r$.

Upper bound for $\|f\|_p$. Let $x \le b_{n+J}$. Then $f(x) = x^\mu G(x)$, where $G(x)$ is the product of the other terms. For $k \le p$,

$$f^{(k)}(x) = \sum_{i=0}^{k} \binom{k}{i} \mu \dots (\mu - i + 1) x^{\mu - i} G^{(k-i)}(x)$$

and so

$$|f^{(k)}(x)| \le \sum_{i=0}^{k} \binom{k}{i} \mu \dots (\mu - i + 1) b_{n+J}^{\mu - i} |G^{(k-i)}(x)|.$$

By Lemma 3, $|G^{(k-i)}(x)| \le C(J, \mu, k-i) b_n^{\mu J - k + i}$. Since $b_{n+J}^{-i} b_n^i$ attains its maximum value at $i = k$, we obtain

$$|f^{(k)}(x)| \le \lambda_n b_{n+J}^{\mu - k} \tag{9}$$

with $\lambda_n = C_p b_n^{\mu J}$, where C_p does not depend on n.

If $x \in I_l$, $n \le l \le n + J - 1$, then writing $f(x) = (x - a_l)^\mu H(x)$ where $H(x)$ is the product of the other terms and arguing as above yields

$$\begin{aligned} |f^{(k)}(x)| &\le \sum_{i=0}^{k} \binom{k}{i} \mu \dots (\mu - i + 1) |x - a_l|^{\mu - i} |H^{(k-i)}(x)| \\ &\le \sum_{i=0}^{k} \binom{k}{i} \mu \dots (\mu - i + 1) \psi_l^{\mu - i} C(J, \mu, k - i) b_n^{\mu J - k + i}. \end{aligned}$$

Since $\psi_l^{-i} b_n^i$ attains its maximum value at $i = k$, we obtain

$$|f^{(k)}(x)| \le \lambda_n \psi_l^{\mu - k}. \tag{10}$$

We therefore have

$$|f^{(k)}(x)| \le \lambda_n \psi_n^{\mu-p} \le \lambda_n \psi_n^Q \quad \text{if } x \le b_{n+J} \text{ or } x \in I_l,\ n \le l \le n+J-1.$$

Next we consider

$$A_p = \frac{|(R_y^p f)^{(i)}(x)|}{|x-y|^{p-i}}, \quad x, y \in K, \quad x \ne y, \quad i \le p.$$

If $x, y \le b_{n+J}$ or $x, y \in I_l$ $(n \le l \le n+J-1)$, then by (7) we have

$$R_y^p f^{(i)}(x) = (f^{(p)}(\xi) - f^{(p)}(y)) \frac{(x-y)^{p-i}}{(p-i)!},$$

where $0 < \xi < b_{n+J}$ or $\xi \in I_l$. Therefore, in this case $A_p \le 2\lambda_n \psi_n^Q$.

If $x \in I_l$ and $y \in I_m$ $(n \le l, m \le n+J-1)$, then

$$|x-y| \ge \max\{h_l, h_m\} \ge \max\{\psi_l, \psi_m\},$$

and from (10) we see that

$$\begin{aligned} A_p &\le \frac{|f^{(i)}(x)|}{|x-y|^{p-i}} + \sum_{j=i}^{p} \frac{|f^{(j)}(y)|}{(j-i)!} \frac{|x-y|^{j-i}}{|x-y|^{p-i}} \\ &\le \frac{\lambda_n \psi_l^{\mu-i}}{\psi_l^{p-i}} + \sum_{j=i}^{p} \frac{\lambda_n \psi_m^{\mu-j}}{(j-i)!} \frac{1}{\psi_m^{p-j}} \\ &\le \lambda_n \psi_l^{\mu-p} + e\lambda_n \psi_m^{\mu-p} \le 4\lambda_n \psi_n^{\mu-p} \le 4\lambda_n \psi_n^Q. \end{aligned}$$

Clearly, the same estimate holds if $l \ge n$ and $m \le n-1$ or $m \ge n$ and $l \le n-1$.

If $x \le b_{n+J}$ and $y \in I_m$ with $n \le m \le n+J-1$, then by hypothesis $|x-y| \ge h_{n+J-1} \ge b_{n+J}^Q$ and so (9) implies that

$$\frac{|f^{(i)}(x)|}{|x-y|^{p-i}} \le \lambda_n \frac{b_{n+J}^{Q(p+1)-i}}{b_{n+J}^{Q(p-i)}} \le \lambda_n b_{n+J}^Q \le \lambda_n \psi_n^Q.$$

Similarly, since $|x-y| \ge h_m \ge \psi_m$, for $i \le j \le p$ we have

$$\frac{|f^{(j)}(y)|}{|x-y|^{p-j}} \le \lambda_n \psi_m^{\mu-p} \le \lambda_n \psi_n^Q.$$

Thus, by (4), $A_p \le 4\lambda_n \psi_n^Q$. The case $y \le b_{n+J}$ and $x \in I_m$, $n \le m \le n+J-1$, can be treated in exactly the same way. If $x \in I_l$ $(l \le n-1)$ or $y \in I_m$ $(m \le n-1)$, then the value of A_p may only be reduced.

Hence we have that $\|f\|_p \le 5\lambda_n \psi_n^Q \le \psi_n$ for $n \ge n_p$, since $\lambda_n \to 0$ and $Q \ge 1$.

Upper bound for $\|f\|_r$. By Lemma 3, $|f^{(k)}(x)| \le C(J+1, \mu, k) b_n^{q-k}$ for $k \le q$ and 0 otherwise. Thus,

$$|f|_r \le \max_{k \le q} C(J+1, \mu, k) = C_q.$$

Clearly $R_y^r f(x) \equiv 0$ when $x, y \le b_n$. If either $x \ge a_{n-1}$ or $y \ge a_{n-1}$ then, since $|x-y| \ge h_{n-1}$, by (4) we have

$$\frac{|(R_y^r f)^{(i)}(x)|}{|x-y|^{r-i}} \le |f|_r\left(1+\sum_{j=i}^{r}\frac{1}{(j-i)!}\right)\frac{1}{|x-y|^r} \le 4C_q h_{n-1}^{-r}.$$

Thus

$$\|f\|_r \le 5C_q h_{n-1}^{-r}.$$

Now, replacing f by f_n in D_φ, we obtain

$$q! \le \varphi^m(t)\psi_n + \frac{C}{t}5C_q h_{n-1}^{-r} \le \varphi^m(t)\psi_n + \frac{1}{th_{n-1}^{r+1}}$$

for large enough n and arbitrary t. Let $t = h_{n-1}^{-r-1}$. Since $q \ge 2$ we obtain $\psi_n \ge \varphi^{-m}(h_{n-1}^{-r-1})$, and it follows that the asymptotic inequality (8) holds with $M = \max\{m, r+1\}$. Increasing the value of M if necessary, we get (8) for all n.

Sufficiency. Let $p = 0$ and $R = 2MQ$, where Q is taken from (3). For a given $q \ge 1$, let $r = 2q$ and $m = Mq + 1$. It is enough to prove the implication

$$\|f\|_0 \le \tau,\ \|f\|_r \le t \Rightarrow \|f\|_q \le 1 \quad \text{where } \tau = \frac{1}{\varphi^m(t^R)}$$

for each fixed $f \in \mathcal{E}(K)$. In fact, $\tau U_0 \cap tU_r \subset U_q$ implies

$$\|f\|_q \le \max\left\{\frac{\|f\|_0}{\tau}, \frac{\|f\|_r}{t}\right\} \le \varphi^m(t^R)\|f\|_0 + \frac{1}{t}\|f\|_r$$

(i.e. D_φ) since R does not depend on q (see e.g. [9]).

Fix any t such that $t^2 > 1/b_1$. Find n such that $b_{n+1} \le t^{-2} < b_n$. Then $h_n \ge b_n^Q > 1/t^{2Q}$ and, by the hypothesis, we have

$$\psi_{n+1} \ge \delta_0 \overset{\text{def}}{=} \frac{1}{\varphi^M(t^{2MQ})}.$$

It is clear that

$$\delta_0 t^2 \le 1 \quad \text{and} \quad \frac{\tau}{\delta_0^q} \le \frac{1}{t}. \tag{11}$$

Let us first estimate

$$B := |f^{(k)}(z)|t^{2(q-k)}, \quad z \in K,\ k \le q.$$

If $z \ge a_{n+1}$, then one can apply Lemma 1 and (11) and get

$$B \le (C_1\delta_0^{-k}\tau + C_2\delta_0^{r-k}t)t^{2(q-k)} = C_1(\delta_0 t^2)^{q-k}\delta_0^{-q}\tau + C_2 t^{2(q-r)+1}.$$

Thus,

$$B \le C_1 t^{-1} + C_2 t^{1-2q}. \tag{12}$$

If $z \le b_{n+2}$ then we consider the Taylor expansion of $f^{(k)}$ at the point $a = a_{n+1}$:

$$f^{(k)}(z) = \sum_{i=k}^{q+1} f^{(i)}(a)\frac{(z-a)^{i-k}}{(i-k)!} + (R_a^{q+1}f)^{(k)}(z).$$

We apply Lemma 1 to the terms $f^{(i)}(a)$. Since $|z-a| \le a \le b_{n+1} < t^{-2}$, we have

$$\begin{aligned}
B &\le \sum_{i=k}^{q+1} (C_1 \delta_0^{-i} \tau + C_2 \delta_0^{r-i} t) \frac{t^{2(q-i)}}{(i-k)!} + \|f\|_{q+1} t^{-2(q+1-k)} \\
&\le \sum_{i=k}^{q+1} (C_1 (\delta_0 t^2)^{q-i} \delta_0^{-q} \tau + C_2 t^{2(q-r)+1}) \frac{1}{(i-k)!} + t^{-1} \\
&\le (C_1 t^{-1} + C_2 t^{1-2q}) e + t^{-1}
\end{aligned}$$

Thus, for some C_5 we have

$$|f^{(k)}(z)| t^{2(q-k)} \le C_5/t, \quad z \in K, \ k \le q. \tag{13}$$

It follows immediately that

$$|f|_q \le C_5/t.$$

Next we estimate

$$A_q = \frac{|(R_y^{(q)} f)^{(i)}(x)|}{|x-y|^{q-i}}, \quad x, y \in K, \ x \ne y, \ i \le q.$$

If $|x-y| \le t^{-2}$, then

$$(R_y^q f)^{(i)}(x) = (R_y^{q+1} f)^{(i)}(x) + f^{(q+1)}(y) \frac{(x-y)^{q+1-i}}{(q+1-i)!},$$

and it follows that

$$A_q \le (\|f\|_{q+1} + |f|_{q+1}) |x-y| \le 2/t.$$

If $|x-y| > t^{-2}$, then by (4) and (13) we have

$$\begin{aligned}
A_q &\le |f^{(i)}(x)| |x-y|^{i-q} + \sum_{k=i}^{q} |f^{(k)}(y)| \frac{|x-y|^{k-q}}{(k-i)!} \\
&\le |f^{(i)}(x)| t^{2(q-i)} + \sum_{k=i}^{q} |f^{(k)}(y)| \frac{t^{2(q-k)}}{(k-i)!} \le \frac{C_5(e+1)}{t}.
\end{aligned}$$

Therefore, for large enough t we obtain $\|f\|_q \le 1$ and so the space $\mathcal{E}(K)$ has the property D_φ. □

REMARK. In the proof of sufficiency, we did not use the restriction about (J_n). In the particular case $\varphi(t) = t$, condition (8) implies that the space $\mathcal{E}(K)$ has the property $\underline{DN} = \Omega_2$. However our proof above gives also the next corollary.

COROLLARY 1. *If there exists an $M > 0$ such that $\psi_n \ge h_{n-1}^M$ for all n, then $\mathcal{E}(K)$ has property $DN = D_1$.*

Proof. It is sufficient to show that, for an arbitrary $f \in \mathcal{E}(K)$ and large enough t,

$$\|f\|_0 \le t^{-Rq}, \ \|f\|_r \le t^q \Rightarrow \|f\|_q \le 1,$$

where R does not depend on q. With $R = 2MQ + 1$, $r = 2q$, $\delta_0 = t^{-2MQ}$, and $\tau = t^{-Rq}$, we see that (11) holds and the proof of the sufficiency of Theorem 1 can be repeated. □

COROLLARY 2. *Let there exist J such that $J_n \leq J$ for all n. Then the following are equivalent:*

(a) *$\mathcal{E}(K)$ has $DN = D_1$;*
(b) *$\mathcal{E}(K)$ has $\underline{DN} = \Omega_2$;*
(c) *$\exists M > 0 : \psi_n \geq h_{n-1}^M$ for all n.*

Proof. (a) trivially implies (b), and (b) is equivalent to (c) by Theorem 1. □

THEOREM 2. *Let $\lim_{n\to\infty} J_n = \infty$. Then $\mathcal{E}(K)$ has D_φ if and only if the following condition holds:*

$$\exists M > 0 : \psi_n \geq \varphi^{-M}(h_n^{-M}) \; \forall n. \tag{14}$$

Proof.

Necessity. By D_φ we have p. Let $q = p + 1$, and let

$$f = f_n = \begin{cases} (x - a_n)^q/q! & \text{if } x \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily checked that $\|f\|_p \leq 4\psi_n$, $\|f\|_q \geq 1$, and $\|f\|_r \leq 4h_n^{-r}$. Now in D_φ we let $t = 8Ch_n^{-r}$ and obtain $1 \leq 8\varphi^m(t)\psi_n$ which gives (14).

Sufficiency. Suppose again that M in (14) is a natural number. Let Q be as in (3). Let $p = 0$, $R = 2MQ$, and, for any $q \geq 1$, let $r = 2q$, $m = MQq + 1$, and $\tau = \varphi^{-m}(t^R)$. We will show that, for any $f \in \mathcal{E}(K)$ with $\|f\|_0 \leq \tau$ and $\|f\|_r \leq t$ with t large enough, we have $\|f\|_q \leq 1$.

Given t large enough, find n such that $b_{n+1} \leq t^{-2} < b_n$. Then $\psi_n \geq \varphi^{-M}(t^R) \stackrel{\text{def}}{=} \delta_0$ and

$$\delta_0 t^2 \leq 1, \quad \delta_0^{-Qq}\tau \leq t^{-1}. \tag{15}$$

Now we can repeat (with some modifications) the proof of Theorem 1. Consider the same B. If $z \geq a_n$ then, applying Lemma 1 and (15), we get (12). In order to find an upper bound for B when $z \leq b_{n+1}$, we consider the following special point $x_0 = b_{n+r+1}$. Choosing $x_k = b_{n+r+1-k}$, $k = 0, 1, \dots, r$, we can use Lemma 2. Since $J_n \to \infty$ as $n \to \infty$, we have $h = x_1 - x_0 > h_{n+r} \geq b_{n+r}^Q > \psi_n^Q$ for large enough n. Thus $h^{-1} \leq \delta_0^{-Q}$. On the other hand, $H = x_r - x_{r-1} < b_{n+1} \leq t^{-2}$. Therefore, for $i \leq q + 1$,

$$\begin{aligned} |f^{(i)}(x_0)|t^{2(q-i)} &\leq (C_3\delta_0^{-Qi}\tau + C_4 t^{-2(r-i)} t)t^{2(q-i)} \\ &\leq C_3(\delta_0^Q t^2)^{q-i}\delta_0^{-Qq}\tau + C_4 t^{1-2(r-q)} \leq (C_3 + C_4)t^{-1}. \end{aligned} \tag{16}$$

Now for any $z \in K$, $z \leq b_{n+1}$, one can use the expansion

$$f^{(k)}(z) = \sum_{i=k}^{q+1} f^{(j)}(x_0)\frac{(z - x_0)^{i-k}}{(i - k)!} + (R_{x_0}^{q+1} f)^{(k)}(z).$$

By using $|z - x_0| < b_{n+1} \leq t^{-2}$ and (16), we get

$$
\begin{aligned}
B &\le \sum_{i=k}^{q+1} |f^{(i)}(x_0)| \frac{|z-x_0|^{i-k}}{(i-k)!} t^{2(q-k)} + \|f\|_{q+1} |z-x_0|^{q+1-k} t^{2(q-k)} \\
&\le \sum_{i=k}^{q+1} |f^{(i)}(x_0)| \frac{t^{-2(i-k)+2(q-k)}}{(i-k)!} + \|f\|_{q+1} t^{-2(q+1-k)+2(q-k)} \\
&\le \frac{C_3 + C_4}{t} e + \frac{t}{t^2}.
\end{aligned}
$$

Therefore, for some C_6,

$$
|f^{(k)}(z)| t^{2(q-k)} \le C_6/t, \quad z \in K, \ k \le q,
$$

which trivially gives the bound for $|f|_q$. Arguing as in Theorem 1 with the same A_q, we see that $\|f\|_q \le 1$ if $t \ge t_0$. Thus the space $\mathcal{E}(K)$ has property D_φ. □

COROLLARY 1. *If $\mathcal{E}(K)$ has the property $DN = D_1$ then there is an $M > 0$ such that $\psi_n \ge h_n^M$ for all n.*

COROLLARY 2. *Let $J_n \to \infty$ as $n \to \infty$. Then the following are equivalent:*

(a) *$\mathcal{E}(K)$ has property $DN = D_1$;*
(b) *$\mathcal{E}(K)$ has property $\underline{DN} = \Omega_2$;*
(c) *$\exists M > 0 : \psi_n \ge h_n^M$ for all n;*
(d) *$\exists M > 0 : \psi_n \ge h_{n-1}^M$ for all n.*

Proof. (a) trivially implies (b), which is equivalent to (c) by Theorem 2. To prove (c) $\Rightarrow$ (d), observe that $b_n > \psi_{n-1}$ since $J_n \to \infty$. Hence

$$
\psi_n \ge h_n^M \ge b_n^{MQ} > \psi_{n-1}^{MQ} \ge h_{n-1}^{M^2 Q}.
$$

Finally, (d) $\Rightarrow$ (a) is exactly Corollary 1 to Theorem 1. □

In [21] Tidten introduced the following geometric property of compact sets in $\mathbb{R}$.

DEFINITION. Let $\alpha \ge 1$. A compact set $K \subset \mathbb{R}$ is said to belong to the class (α) if there exist $\delta_0 > 0$ and $C > 0$ such that, for any point $y \in K$, there is a sequence (x_j) in K with the following properties:

(i) $|y - x_j| \downarrow 0$,
(ii) $|y - x_1| \ge \delta_0$,
(iii) $C|y - x_{j+1}| \ge |y - x_j|^\alpha$ for all j.

In our case we have the following.

PROPOSITION 1. *There exists an $\alpha \ge 1$ such that $K \in (\alpha)$ if and only if the following condition holds:*

$$
\exists M > 0 : \psi_n \ge h_{n-1}^M \ \forall n.
$$

Proof. Suppose $K \in (\alpha)$. Find n_0 such that $b_n < \delta_0$ for all $n \ge n_0$. Given $n \ge n_0$, let (x_j) be the corresponding sequence for $y = b_n$. Then $x_1 \ge a_{n-1}$. Let x_{i+1} be the first term of this sequence in I_n. Then

$$C\psi_n \geq C|y - x_{i+1}| \geq |y - x_i|^\alpha > h_n^\alpha \geq b_n^{\alpha Q}.$$

On the other hand, let x_{k+1} be the first term of the same sequence in $[0, b_n] \cap K$. Since $x_k \geq a_{n-1}$, we have

$$Cb_n \geq C|y - x_{k+1}| \geq |y - x_k|^\alpha \geq h_{n-1}^\alpha.$$

Thus we have that $C^{\alpha Q+1}\psi_n \geq h_{n-1}^{\alpha^2 Q}$.

For the proof of sufficiency we do not need condition (3). Without loss of generality, we may assume that $M \geq 1$. Let $\alpha = M$, $\delta_0 = \frac{1}{2}\psi_1$, and $C = 2 \cdot 3^M$. Let $y \in K$ be given. If $y = 0$, we see that the sequence $(x_j) = (b_j)$ is suitable because

$$b_j = b_{j+1} + h_j + \psi_j \leq b_{j+1} + 2h_j \leq b_{j+1} + 2\psi_{j+1}^{1/M} \leq 3b_{j+1}^{1/M}.$$

If $y \in I_n$ for some n, then let $x_j = b_j$ $(j = 1, 2, \ldots, n-1)$ and let x_n be the endpoint of I_n that is farther from y. Then $(x_j)_{j\geq n+1} \subset I_n$ can be constructed trivially, and for $j = 1, \ldots, n-2$ we have $|y - x_{j+1}| := d > \psi_{j+1}$. On the other hand,

$$|y - x_j| = d + h_j + \psi_j \leq d + 2h_j \leq d + 2\psi_{j+1}^{1/M} \leq 3d^{1/M}.$$

In turn,

$$|y-x_n| \geq \psi_n/2 \quad \text{and} \quad |y-x_{n-1}| \leq \psi_n+h_{n-1}+\psi_{n-1} \leq \psi_n+2h_{n-1} \leq 3\psi_n^{1/M}.$$

Thus $K \in (M)$. □

We note that the restriction $h_n \geq b_n^Q$ cannot be omitted for the necessity part. One can construct a compact set $K \in (1)$ such that

$$\limsup_{n\to\infty} \frac{\ln \psi_n}{\ln h_{n-1}} = \infty.$$

For an arbitrary compact set $K \subset \mathbb{R}$, Tidten [21] has proved that

$$K \in (1) \Rightarrow \mathcal{E}(K) \text{ has } DN \Rightarrow K \in (\alpha) \text{ for some } \alpha \geq 1.$$

In our case, we have the following criterion.

THEOREM 3. *Let the compact set $K \subset \mathbb{R}$ be as in Section I. Then the following are equivalent:*

(a) *$\mathcal{E}(K)$ has $DN = D_1$;*
(b) *$\exists \alpha \geq 1 : K \in (\alpha)$;*
(c) *$\exists M > 0 : \psi_n \geq h_{n-1}^M$ for all n.*

Proof. It follows from [21] that (a) implies (b). By Proposition 1, (b) is equivalent to (c). By Corollary 1 to Theorem 1, we have that (c) implies (a). □

Because the property DN is equivalent (on the one hand) to existence of a linear continuous extension operator $\mathcal{E}(K) \to C^\infty(\mathbb{R})$ [20] and (on the other hand) to the isomorphism of $\mathcal{E}(K)$ to the space of rapidly decreasing sequences s [25], conditions (b) and (c) of Theorem 3 are geometric characterizations of the above properties for the space $\mathcal{E}(K)$ when K satisfies (2).

III. Linear Topological Invariant β

The use of geometrical linear topological invariants as a tool for the isomorphic classification of nonnormed Fréchet spaces and linear topological spaces first appeared in the works of Pełczynski [19] and Kolmogorov [14]. Later, Bessaga, Pełczyński, and Rolewicz [2] and Mitiagin [16] considered the approximative and diametral dimensions that are more convenient for regular Köthe spaces. In [30; 32], Zahariuta introduced some general characteristics as generalizations of Mitiagin's invariants in [17; 18], and in [33; 34] considered new geometrical invariants in terms of synthetic neighborhoods. Further use of geometrical invariants appeared in [3; 4; 5; 6; 7; 10; 11; 13; 29; 34]. Interpolational linear topological invariants such as DN and Ω and their variations were introduced and used by Vogt and Wagner in [23; 24; 27; 28] to characterize the subspaces and quotient spaces of stable power series spaces.

Let E be a linear space, and let U and V be two absolutely convex sets in E. Following ideas of Zahariuta about synthetic neighborhoods [33], we consider

$$\tilde{\beta}(U, V) = \min\{\dim L : U \subset V + L\}.$$

It is clear that if $U_1 \subset U_2$ and $V_1 \supset V_2$, then $\tilde{\beta}(U_1, V_1) \leq \tilde{\beta}(U_2, V_2)$. Now let X be a Fréchet space with a fundamental system of neighborhoods (U_p), and let $t, \tau \in \mathbb{R}_+$. In what follows, $t \to \infty$ and $\tau = \tau(t) \to 0$. Given $0 \leq p < q < r$, we set $\tilde{U} = \tau U_p \cap t U_r$ and define

$$\beta(\tau, t;\ U_p, U_q, U_r) = \tilde{\beta}(\tilde{U}, U_q) = \min\{\dim L : \tilde{U} \subset U_q + L\}.$$

Proposition 2. *Let X and Y be isomorphic Fréchet spaces with fundamental system of neighborhoods (U_p) and (V_p), respectively. Then*

$$\forall p\ \exists p_1\ \forall q_1\ \exists q\ \forall r\ \exists r_1\ \exists \varepsilon :$$
$$\beta(\tau, t;\ V_{p_1}, V_{q_1}, V_{r_1}) \leq \beta(\tau, t;\ U_p, \varepsilon U_q, U_r) \quad \forall t, \tau > 0.$$

and vice versa.

Proof. Let $T: X \to Y$ be an isomorphism. Then, with the above order of quantifiers and for some $C \geq 1$, we have

$$V_{p_1} \subset CT(U_p), \quad \frac{1}{C}T(U_q) \subset V_{q_1}, \quad V_{r_1} \subset CT(U_r).$$

Then we have

$$\beta(\tau, t;\ V_{p_1}, V_{q_1}, V_{r_1}) \leq \beta(\tau, t;\ CT(U_p),$$
$$\frac{1}{C}T(U_q), CT(U_r)) = \beta\left(\tau, t;\ U_p, \frac{1}{C^2}U_q, U_r\right). \qquad \square$$

When X is a Schwartz space and U_r is precompact with respect to U_q, it is easy to see that β is finite and

$$\beta(\tau, t;\ U_p, U_q, U_r) = |\{n : d_n(\tilde{U}, U_q) > 1\}|,$$

where $d_n(\tilde{U}, U_q)$ denotes the nth Kolmogorov diameter of $\tilde{U}$ with respect to U_q and where, for a set S, by $|S|$ we denote the number of elements of S if it is finite and ∞ if it is an infinite set.

Invariants D_φ and β are closely related as described in the following proposition (for a proof see [8, Prop. 2]).

PROPOSITION 3. *Let X be a Fréchet space with a fundamental system of neighborhoods (U_p), and let φ be as in the definition of property D_φ. Then the following are equivalent:*

(a) *X has the property D_φ;*
(b) $\exists(p, R), \forall q, \exists(r, m, t_0) : \beta(\tau, t;\ U_p, U_q, U_r) = 0$ *for all* $t \geq t_0$, *where* $\tau = \varphi^{-m}(t^R)$.

We will find upper and lower bounds for β when $X = \mathcal{E}(K)$ with K as defined in Section I.

Let p, q, r be such that $0 \leq p < q$ and $qQ < r$, where Q satisfies (2). Let $\delta_0^q = 4(e+1)C_1\tau$ and $\tau = o(t^{-q/(r-q)})$; more precisely,

$$t\delta_0^{r-q}4C_2(e+1) \leq 1.$$

Here, C_1 and C_2 are as in Lemma 1. We choose n_1 and N_1 as follows:

$$n_1 = \min\{n : \psi_n < \delta_0\}, \qquad N_1 = \min\{n : b_n^{r-qQ} \leq 1/4et\}. \tag{17}$$

Then we have

$$\psi_{n_1}^{r-q} < \delta_0^{r-q} \leq \frac{1}{4C_2(e+1)t}, \quad \psi_{n_1-1} \geq \delta_0, \quad b_{N_1}^{r-qQ}4et \leq 1. \tag{18}$$

Assume $n_1 \leq N_1$. For $n = 1, 2, \dots$ and $j = 0, 1, \dots, r$, let

$$e_{nj}(x) = \begin{cases} (x-a_n)^j/j!, & x \in I_n, \\ 0, & x \in K \setminus I_n; \end{cases} \qquad E_{nj}(x) = \begin{cases} x^j/j!, & x \in K \cap [0, b_n], \\ 0, & \text{otherwise on } K. \end{cases}$$

Upper Bound for β

Obviously, if there is a subspace L of dimension m such that $\tilde{U} \subset U_q + L$, then $\beta \leq m$. Let

$$L = \operatorname{span}(\{E_{N_1 j} : j = 0, \dots, r\} \cup \{e_{kj} : k = n_1, n_1+1, \dots, N_1 - 1;\ j = 0, 1, \dots, r\}).$$

(If $n_1 = N_1$, then the second set in the union is empty.) Then $\dim L = (r+1)(N_1 - n_1 + 1)$. Given $f \in \mathcal{E}(K)$, let

$$g = \sum_{j=0}^{r}\left[f^{(j)}(0)E_{N_1 j} + \sum_{k=n_1}^{N_1-1} f^{(j)}(a_k)e_{kj}\right].$$

Clearly $g \in L$. If for each $f \in \tilde{U}$ (i.e., $\|f\|_p \leq \tau$ and $\|f\|_r \leq t$) we show that $\|f - g\|_q \leq 1$, then it will follow that $\tilde{U} \subset U_q + L$ and so $\beta(\tau, t) \leq \dim L$. Now we write $f - g$ more explicitly. Let $z \in I_j$. Then

$$f(z) - g(z) = \begin{cases} f(z) & \text{if } j < n_1, \\ R^r_{a_j} f(z) & \text{if } n_1 \le j < N_1, \\ R^r_0 f(z) & \text{if } N_1 \le j. \end{cases} \tag{19}$$

Up-1. *Upper bound for* $|f - g|_q$. Let $i \le q$, $z \in I_j$, and $j \in \mathbb{N}$.

(1.1) $j \ge N_1$. Then $(f - g)(z) = R^r_0 f(z)$ and so, by (18),

$$|(f-g)^{(i)}(z)| = |(R^r_0 f)^{(i)}(z)| \le \|f\|_r z^{r-i} \le \|f\|_r b_{N_1}^{r-i} \le t b_{N_1}^{r-q} \le 1/2.$$

(1.2) $n_1 \le j \le N_1 - 1$. Then $(f - g)(z) = R^r_{a_j} f(z)$ and so, by (18),

$$|(f-g)^{(i)}(z)| \le \|f\|_r |z - a_j|^{r-i} \le t\psi_j^{r-i} \le t\psi_{n_1}^{r-q} \le 1/2.$$

(1.3) $j < n_1$. Then $g(z) = 0$. Since $\psi_j \ge \psi_{n_1-1} \ge \delta_0$, for $i \ge p$ we may apply Lemma 1 and obtain

$$|f^{(i)}(z)| \le C_1 \delta_0^{-q+p} \tau + C_2 \delta_0^{r-q} t \le \frac{1}{4(e+1)} + \frac{1}{4(e+1)} \le \frac{1}{2}.$$

If $i < p$ then $|f^{(i)}(z)| \le \|f\|_p \le \tau \le \frac{1}{2}$.

Thus $|f - g|_q \le \frac{1}{2}$.

Up-2. *Upper bound for* $A_{q,i} = |(R^q_y(f-g))^{(i)}(x)||x-y|^{i-q}$, $i \le q$.

(2.1) $x, y \le b_{N_1}$. Then $g(x) = T^r_0 f(x)$ and, by (19),

$$\begin{aligned} R^q_y(f-g)(x) &= f(x) - g(x) - T^q_y(R^r_0 f)(x) \\ &= T^r_y f(x) + R^r_y f(x) - T^r_0 f(x) - T^q_y(R^r_0 f)(x). \end{aligned}$$

Now, by (6),

$$T^r_y f(x) - T^r_0 f(x) = T^r_y(R^r_0 f)(x)$$

and so

$$R^q_y(f-g)(x) = R^r_y f(x) + \sum_{k=q+1}^{r} (R^r_0 f)^{(k)}(y) \frac{(x-y)^k}{k!}$$

$$\Rightarrow (R^q_y(f-g))^{(i)}(x) = (R^r_y f)^{(i)}(x) + \sum_{k=q+1}^{r} (R^r_0 f)^{(k)}(y) \frac{(x-y)^{k-i}}{(k-i)!},$$

which implies

$$\begin{aligned} A_{q,i} &\le \left(\|f\|_r |x-y|^{r-i} + \sum_{k=q+1}^{r} \|f\|_r y^{r-k} \frac{|x-y|^{k-i}}{(k-i)!} \right) |x-y|^{i-q} \\ &\le t|x-y|^{r-q} + \sum_{k=q+1}^{r} t y^{r-k} \frac{|x-y|^{k-q}}{(k-i)!} \\ &\le t b_{N_1}^{r-q} \left(1 + \sum_{k=q+1}^{r} \frac{1}{(k-i)!} \right) \le t b_{N_1}^{r-q} e \le \frac{1}{2}. \end{aligned}$$

(2.2) $x, y \in I_j$, with $n_1 \le j \le N_1 - 1$. Then, exactly as in the proof of (2.1), we have

$$R_y^q(f-g)(x) = R_y^r f(x) + \sum_{k=q+1}^{r} (R_{a_j}^r f)^{(k)}(y)\frac{(x-y)^k}{k!}$$

$$\Rightarrow (R_y^q(f-g))^{(i)}(x) = (R_y^r f)^{(i)}(x) + \sum_{k=q+1}^{r} (R_{a_j}^r f)^{(k)}(y)\frac{(x-y)^{k-i}}{(k-i)!}.$$

Since $|x-y| \le \psi_j \le \psi_{n_1}$ and $|y-a_j| \le \psi_{n_1}$, we have, as in (2.1),

$$A_{q,i} \le t\psi_{n_1}^{r-q} e \le 1/2.$$

(2.3) $x, y \in I_j$, with $j < n_1$. Then, by (19) and (7),

$$A_{q,i} = |(R_y^q f)^{(i)}(x)||x-y|^{i-q} = \frac{1}{(q-i)!}|f^{(q)}(\xi) - f^{(q)}(y)|$$

for some $\xi \in I_j$. We may apply Lemma 1 (since $\psi_j \ge \delta_0$) and derive

$$A_{q,i} \le 2(C_1\delta_0^{-q+p}\tau + C_2\delta_0^{r-q}t) \le \frac{1}{e+1} < \frac{1}{2}.$$

Next we consider cases in which x and y lie in different intervals.

(2.4) $x \in I_l$ and $y \in I_m$, with $n_1 \le l, m < N_1$. Then $\psi_l \le |x-y|$ and $\psi_m \le |x-y|$, since $\psi_k \le h_k$ for all k. Hence

$$R_y^q(f-g)(x) = R_{a_l}^r f(x) - T_y^q(R_{a_m}^r f)(x),$$

and this gives

$$\begin{aligned} A_{q,i} &\le t\psi_l^{r-i}|x-y|^{i-q} + \sum_{k=i}^{q} t\psi_m^{r-k}|x-y|^{k-q}\frac{1}{(k-i)!} \\ &= t\left(\frac{\psi_l}{|x-y|}\right)^{q-i}\psi_l^{r-q} + \sum_{k=i}^{q} t\left(\frac{\psi_m}{|x-y|}\right)^{q-k}\psi_m^{r-q}\frac{1}{(k-i)!} \\ &\le t\psi_{n_1}^{r-q}(1+e) < \frac{1}{2}. \end{aligned}$$

(2.5) $x \in I_l$ and $y \in I_m$, where either $n_1 \le l < N_1$ and $N_1 \le m$ or $n_1 \le m < N_1$ and $N_1 \le l$. Proofs are similar, so let us give the proof only for the case $n_1 \le l < N_1$, $N_1 \le m$:

$$\begin{aligned} &R_y^q(f-g)(x) = R_{a_l}^r f(x) - T_y^q(R_0^r f)(x) \\ &\qquad \Rightarrow A_{q,i} \le t\psi_l^{r-i}|x-y|^{i-q} + \sum_{k=i}^{q} t y^{r-k}|x-y|^{k-q}\frac{1}{(k-i)!} \\ &\qquad\qquad \le t\psi_l^{r-q} + \sum_{k=i}^{q} t\frac{b_{N_1}^{r-k}}{h_{N_1-1}^{q-k}}\frac{1}{(k-i)!}, \end{aligned}$$

since $|x-y| \ge h_{N_1-1}$. Now we remember that $h_n \ge b_{n+1}^Q$ for all n and estimate the term inside the summation as

$$\frac{b_{N_1}^{r-k}}{h_{N_1-1}^{q-k}} \le b_{N_1}^{r-k-Q(q-k)} \le b_{N_1}^{r-Qq}. \tag{20}$$

Then

$$A_{q,i} \leq t\psi_{n_1}^{r-q} + tb_{N_1}^{r-Qq}e \leq 1/2.$$

(2.6) $x \in I_l$ and $y \in I_m$, with $l < n_1$ and $n_1 \leq m < N_1$. Then $g(x) = 0$ and $\psi_l \geq \delta_0$. Hence

$$R_y^q(f - g)(x) = f(x) - T_y^q(R_{a_m}^r f)(x)$$

and

$$A_{q,i} \leq |f^{(i)}(x)||x - y|^{i-q} + \sum_{k=i}^{q} t\psi_m^{r-k}|x - y|^{k-q}\frac{1}{(k - i)!}.$$

If $i \leq p$ then

$$A_{q,i} \leq \tau\delta_0^{-q} + t\psi_m^{r-q}e \leq 1/2,$$

since $|x - y| \geq h_{n_1-1} \geq \psi_{n_1-1} \geq \delta_0$ and $\psi_m \leq |x - y|$. If $i > p$ we use Lemma 1;

$$\begin{aligned} A_{q,i} &\leq (C_1\delta_0^{-i+p}\tau + C_2\delta_0^{r-i}t)|x - y|^{i-q} + t\psi_m^{r-q}e \\ &\leq C_1\delta_0^{p-q}\tau + C_2\delta_0^{r-q}t + t\delta_0^{r-q}e \leq 1/2. \end{aligned}$$

(2.7) $x \in I_l$ and $y \in I_m$, with $l < n_1$ and $N_1 \leq m$. Then

$$\begin{aligned} &R_y^q(f - g)(x) = f(x) - T_y^q(R_0^r f)(x) \\ &\qquad \Rightarrow A_{q,i} \leq |f^{(i)}(x)||x - y|^{i-q} + \sum_{k=i}^{q}|(R_0^r f)^{(k)}(y)|\frac{|x - y|^{k-q}}{(k - i)!}. \end{aligned}$$

Now we can treat the first term as in (2.6) and the summation term as in (2.5), yielding again $A_{q,i} \leq \frac{1}{2}$.

(2.8) $x \in I_l$ and $y \in I_m$, with $m < n_1$ and $n_1 \leq l < N_1$. Then $g(y) = 0$, $\psi_m \geq \delta_0$, and $|x - y| \geq \delta_0$. Hence

$$\begin{aligned} &R_y^q(f - g)(x) = R_{a_l}^r f(x) - T_y^q f(x) \\ &\qquad \Rightarrow A_{q,i} \leq t|x - a_l|^{r-i}|x - y|^{i-q} + \sum_{k=i}^{q}|f^{(k)}(y)|\frac{|x - y|^{k-q}}{(k - i)!}. \end{aligned}$$

Now

$$\text{first term} \leq t\frac{\psi_l^{r-i}}{\delta_0^{q-i}} \leq t\frac{\delta_0^{r-i}}{\delta_0^{q-i}} = t\delta_0^{r-q};$$

to estimate the second term, we repeat the arguments of (2.6).

If $i < p$, then $|f^{(k)}(y)||x - y|^{k-q} \leq \tau\delta_0^{-q}$. If $i \geq p$, then $|f^{(k)}(y)||x - y|^{k-q} \leq C_1\delta_0^{p-q}\tau + C_2\delta_0^{r-q}t$. Thus, by (18),

$$A_{q,i} \leq C_1 e\tau\delta_0^{-q} + (C_2 e + 1)t\delta_0^{r-q} \leq 1/2.$$

(2.9) $x \in I_l$ and $y \in I_m$, where $m < n_1$ and $N_1 \leq l$. Then

$$\begin{aligned} &R_y^q(f - g)(x) = R_0^r f(x) - T_y^q f(x) \\ &\qquad \Rightarrow A_{q,i} \leq tx^{r-i}|x - y|^{i-q} + \sum_{k=i}^{q}|f^{(k)}(y)|\frac{|x - y|^{k-q}}{(k - i)!}. \end{aligned}$$

We treat the first term as in (20) and the second term as in (2.8) and obtain, as before, $A_{q,i} \leq \frac{1}{2}$.

(2.10) $x \in I_l$ and $y \in I_m$, where $l, m < n_1$. Then $g(x) = g(y) = 0$, $\psi_m \geq \delta_0$, $\psi_l \geq \delta_0$, and $|x - y| \geq \delta_0$. We have $R_y^q(f - g)(x) = R_y^q f(x)$ and

$$A_{q,i} \leq |f^{(i)}(x)||x - y|^{i-q} + \sum_{k=i}^{q} |f^{(k)}(y)| \frac{|x - y|^{k-i}}{(k - i)!} |x - y|^{i-q}.$$

Now we argue as in (2.8):

$$A_{q,i} \leq \tau \delta_0^{p-q} C_1(e + 1) + t \delta_0^{r-q} C_2(e + 1) \leq 1/2.$$

Therefore,

$$\|f - g\|_q = |f - g|_q + \sup\{ A_{q,i} : i \leq q,\ x, y \in K,\ x \neq y \} \leq 1,$$

which shows that $\tilde{U} \subset U_q + L$. Hence

$$\beta(\tau, t;\ U_p, U_q, U_r) \leq (N_1 - n_1 + 1)(r + 1),$$

where n_1 and N_1 are defined in (17) with the restriction $\tau^{r-q} t^q \to 0$ as $t \to \infty$.

Lower Bound for β

Next we find a lower bound for $\beta(\tau, t;\ U_p, U_q, U_r)$. By Tikhomorov's theorem (see e.g. [16, Prop. 6]), we have that

$$\beta(\tau, t;\ U_p, U_q, U_r) \geq \sup\{ \dim L : 2U_q \cap L \subset \tilde{U} \},$$

where the supremum is taken over all finite-dimensional subspaces L of X. We define

$$L = \operatorname{span}\{ e_{nq} : n_2 \leq n \leq N_2 \},$$

where

$$n_2 = \min\{ n : \psi_n^{q-p} \leq \tau/4(e + 1) \} \quad \text{and} \quad N_2 = \max\{ n : h_n^{r-q} \geq 16/t \}.$$

Then $\dim L = N_2 - n_2 + 1$ or 0 if $N_2 < n_2$. We show that $2U_q \cap L \subset \tilde{U}$; it will follow that $\beta(\tau, t;\ U_p, U_q, U_r) \geq N_2 - n_2 + 1$. Let $f \in 2U_q \cap L$ be arbitrary. Then $f = \sum_{k=n_2}^{N_2} \alpha_k e_{kq}$; that is,

$$f(x) = \begin{cases} \alpha_k((x - a_k)^q / q!) & \text{if } x \in I_k,\ n_2 \leq k \leq N_2, \\ 0 & \text{otherwise,} \end{cases}$$

and $|\alpha_k| = |f^{(q)}(a_k)| \leq \|f\|_q \leq 2$. We will show that $f \in \tilde{U}$—in other words, that $\|f\|_p \leq \tau$ and $\|f\|_r \leq t$.

Low-3. *Bound for $\|f\|_p$.* Here $A_{p,i} = |(R_y^p f)^{(i)})(x)||x - y|^{i-p}$, $i \leq p$.

(3.1) Let $z \in I_k$ with $n_2 \leq k \leq N_2$ and $i \leq p$. Then

$$f^{(i)}(z) = \frac{(z - a_k)^{q-i}}{(q - i)!} \alpha_k \Rightarrow |f^{(i)}(z)| \leq \frac{\psi_k^{q-i}}{(q - i)!} 2 \leq 2 \frac{\psi_{n_2}^{q-p}}{(q - p)!} \leq \frac{\tau}{2}.$$

Thus $|f|_p \leq \tau/2$.

In order to estimate $A_{p,i}$ we must consider several cases.

(3.2) $x, y \in I_k$, with $n_2 \le k \le N_2$. Then, by (7), we have

$$A_{p,i} \le |f^{(p+1)}(\eta)||x-y| \le 2\frac{(\eta - a_k)^{q-p-1}}{(q-p-1)!}|x-y| \le 2\psi_k^{q-p} \le \frac{\tau}{2}.$$

(3.3) $x \in I_l$ and $y \in I_m$, with $n_2 \le l, m \le N_2$. Then

$$R_y^p f(x) = \alpha_l \frac{(x-a_l)^q}{q!} - \alpha_m \sum_{k=0}^{p} \frac{(x-y)^k}{k!}\frac{(y-a_m)^{q-k}}{(q-k)!}$$

$$\Rightarrow A_{p,i} \le \frac{2}{(q-i)!}|x-a_l|^{q-i}|x-y|^{i-p} + \sum_{k=i}^{p} \frac{2}{(k-i)!\,(q-k)!}|y-a_m|^{q-k}|x-y|^{k-p}$$

$$\le \frac{2}{(q-i)!}\left(\frac{|x-a_l|}{|x-y|}\right)^{p-i}|x-a_l|^{q-p} + \sum_{k=i}^{p} \frac{2}{(k-i)!\,(q-k)!}\left(\frac{|y-a_m|}{|x-y|}\right)^{p-k}|y-a_m|^{q-p}.$$

Since $|x-a_l| \le \psi_l \le h_l \le |x-y|$ and $|y-a_m| \le \psi_m \le |x-y|$, we have

$$A_{p,i} \le \frac{\psi_l^{q-p}}{(q-i)!}2 + 2\psi_m^{q-p}\frac{1}{(q-p)!}e \le 2(e+1)\psi_{n_2}^{q-p} \le \frac{\tau}{2}.$$

The cases $x \notin \operatorname{supp} f$ or $y \notin \operatorname{supp} f$ can be treated exactly as in (3.3).

Low-4. *Bound for* $\|f\|_r$. Here $A_{r,i} = |(R_y^r f)^{(i)}(x)||x-y|^{i-r}$, $i \le r$, but actually $i \le q$ since $f^{(q+1)} \equiv 0$.

(4.1) $|f|_r = |f|_q \le 2 \le t/2$.

Next we consider the bound for $A_{r,i}$.

(4.2) $x, y \in I_k$; then $R_y^r f \equiv 0$.

(4.3) $x \in I_l$ and $y \in I_m$, with $n_2 \le l, m \le N_2$. Then

$$(R_y^r f)^{(i)}(x) = \alpha_l \frac{(x-a_l)^{q-i}}{(q-i)!} - \alpha_m \sum_{k=i}^{q} \frac{(y-a_m)^{q-k}}{(q-k)!}\frac{(x-y)^{k-i}}{(k-i)!}$$

$$\Rightarrow A_{r,i} \le 2\left(\frac{|x-a_l|}{|x-y|}\right)^{q-i}|x-y|^{q-r} + 2\sum_{k=i}^{q}\left(\frac{|y-a_m|}{|x-y|}\right)^{q-k}\frac{|x-y|^{q-r}}{(k-i)!\,(q-k)!}.$$

We have $|x-y| \ge h_l \ge h_{N_2}$ and $|x-a_l| \le \psi_l \le h_l \le |x-y|$ and similarly $|y-a_m| \le |x-y|$. Thus

$$A_{r,i} \le 2|x-y|^{q-r}(1+e) \le \frac{8}{h_{N_2}^{r-q}} \le \frac{t}{2}.$$

If $x \notin \operatorname{supp} f$ or $y \notin \operatorname{supp} f$, then $A_{r,i}$ may only be reduced.

Thus $\|f\|_p \le \tau$ and $\|f\|_r \le t$, which implies that $f \in \tilde{U}$ and so

$$\beta(\tau, t;\ U_p, U_q, U_r) \ge N_2 - n_2 + 1.$$

EXAMPLE. In this part we give an example of a continuum of spaces $\mathcal{E}(K_\lambda)$ which cannot be distinguished by the linear topological invariant D_φ but which can be shown to be pairwise nonisomorphic by means of the linear topological invariant β.

Let

$$\psi(\tau) = \exp(-1/\tau), \quad 0 < \tau \le 1.$$

Given $\lambda > 1$ and $n > e^3$, we define

$$b_n^{(\lambda)} = \exp(-(\ln n)^\lambda), \quad \psi_n^{(\lambda)} = \exp(-\exp(\ln n)^\lambda) = \psi(b_n^{(\lambda)}),$$
$$a_n^{(\lambda)} = b_n^{(\lambda)} - \psi_n^{(\lambda)}.$$

We denote the corresponding Whitney space by $\mathcal{E}(K_\lambda)$. To simplify notation, we will omit the superscript (λ) except where confusion would result. It can be checked that

$$b_n - b_{n+1} > \lambda \frac{(\ln n)^{\lambda-1}}{n+1} > b_{n+1}^2 \quad \text{and} \quad \psi_n < \frac{1}{2} b_{n+1}^2.$$

Thus $h_n = b_n - b_{n+1} - \psi_n > \frac{1}{2} b_{n+1}^2 \ge b_{n+1}^3$, and (2) is valid with $Q = 3$. It is also clear that $\lim_{n\to\infty} J_n = \infty$.

Now assume that the space $\mathcal{E}(K_\lambda)$ has property D_φ for some φ. Then, for some $M > 0$, we have $\psi_n \ge \varphi^{-M}(h_n^{-M})$; that is,

$$\exp\exp(\ln n)^\lambda \le \varphi^M\left(\frac{1}{(h_n^\lambda)^M}\right).$$

Let $\mu > 1$ be given. Given j, we find $n = n(j)$ such that

$$(\ln n)^\lambda \le (\ln j)^\mu < (\ln(n+1))^\lambda.$$

Then, using $(\ln(n+2))^\lambda \le 2^\lambda (\ln n)^\lambda$, we have

$$\begin{aligned}
\frac{1}{\psi_j^{(\mu)}} &= \exp\exp(\ln j)^\mu \le \exp\exp(\ln(n+1))^\lambda \\
&\le \varphi^M\left(\frac{1}{(h_{n+1}^{(\lambda)})^M}\right) \le \varphi^M\left(\frac{1}{(b_{n+2}^{(\lambda)})^{3M}}\right) \\
&= \varphi^M\Big(\exp 3M(\ln(n+2))^\lambda\Big) \le \varphi^M\Big(\exp N(\ln n)^\lambda\Big) \\
&\le \varphi^N\Big(\exp N(\ln j)^\mu\Big) = \varphi^N\left(\frac{1}{(b_j^{(\mu)})^N}\right) \le \varphi^N\left(\frac{1}{(h_j^{(\mu)})^N}\right),
\end{aligned}$$

where $N = 3M2^\lambda$. For a given φ we thus have that the space $\mathcal{E}(K_\lambda)$ has property D_φ if and only if the space $\mathcal{E}(K_\mu)$ has property D_φ.

Next we show that the spaces $\{\mathcal{E}(K_\lambda) : \lambda > 1\}$ are pairwise nonisomorphic; as our tool we shall use the invariant β with $\tau = \tau(t) = 1/\varphi(t)$, where $\varphi(t) =$

$\exp(\ln t)^2$. Suppose that $\mathcal{E}(K_\lambda)$ is isomorphic to $\mathcal{E}(K_\mu)$ for $\lambda < \mu$. Then, by Proposition 2, $\forall p\ \exists p_1\ \forall q_1\ \exists q\ \forall r\ \exists (r_1, C)$ such that

$$\beta(\tau, t, U_{p_1}^{(\lambda)}, U_{q_1}^{(\lambda)}, U_{r_1}^{(\lambda)}) \leq \beta(C\tau, Ct, U_p^{(\mu)}, U_q^{(\mu)}, U_r^{(\mu)}, \tag{21}$$

where $(U_p^{(\lambda)})$ and $(U_p^{(\mu)})$ denote the neighborhood bases in $\mathcal{E}(K_\lambda)$ and $\mathcal{E}(K_\mu)$, respectively.

We take $p = 0$, $q_1 = p_1 + 2$, $r = 3q + 1$ and estimate $N_1 = N_1(\mu, Ct)$, $n_2 = n_2(\lambda, \tau)$, $N_2 = N_2(\lambda, t)$. For large enough t one has

$$N_1 = \min\{n : \ln(4eCt) \leq (\ln n)^\mu\} \leq \exp(2\ln t)^{1/\mu},$$

$$n_2 = \min\{n : 4(e+1)\exp(\ln t)^2 \leq \exp 2\exp(\ln n)^\lambda\} \leq \exp(\ln(\ln t)^2)^{1/\lambda}.$$

Since $h_n \geq b_{n+1}^3 \geq b_n^{3\cdot 2^\lambda}$, we have

$$N_2 \geq \max\left\{n : b_n^{3r_1\cdot 2^\lambda} \geq \frac{16}{t}\right\} \geq \exp\frac{1}{2}\left(\frac{1}{6r_1}\ln t\right)^{1/\lambda}.$$

Applying the bounds of the corresponding functions β in (21) yields

$$\exp\frac{1}{4}\left(\frac{1}{6r_1}\ln t\right)^{1/\lambda} < N_2 - n_2 \leq (N_1+1)(r+1) \leq 2rN_1 < 2r\exp(2\ln t)^{1/\mu},$$

which is impossible for large t when $1 < \lambda < \mu$ are fixed constants.

Remark. The problem of the existence of basis in the space $\mathcal{E}(K)$ is still open. However, in the space $\mathcal{E}_0(K)$ of functions vanishing at zero, a basis was constructed in [12] under the hypothesis (2). Our results are in accordance with [13] and [1], where the problem of isomorphic classification of Köthe representations of $\mathcal{E}_0(K)$ is considered. On the other hand, in the case of C^∞-functions defined on a domain with a sharp point, the compound invariant is not more refined than the property D_φ (see [8]).

References

[1] B. Arslan, A. Goncharov, and M. Kocatepe, *Sequence space representations of spaces of Whitney functions* (to appear).

[2] C. Bessaga, A. Pełczyński, and S. Rolewicz, *On diametral approximative dimension and linear homogenity of F-spaces,* Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 9 (1961), 677–683.

[3] P. A. Chalov, P. B. Djakov, and V. P. Zahariuta, *Compound invariants and embeddings of Cartesian products,* preprint.

[4] P. A. Chalov and V. P. Zahariuta, *On linear topological invariants,* Manuscript no. 5941-85, deposited at VINITI, 1985 (Russian).

[5] ———, *On linear topological invariants on some class of families of Hilbert spaces,* Manuscript no. 3862-B86, deposited at VINITI, 1986 (Russian).

[6] P. B. Djakov and V. P. Zahariuta, *On Dragilev type power Köthe spaces,* Studia Math. 120 (1996), 219–234.

[7] A. Goncharov, *Isomorphic classification of spaces of infinitely differentiable functions,* dissertation, Rostov Univ., 1986 (Russian).

[8] ———, *Compound invariants and spaces of C^∞ functions,* Linear Topol. Spaces Complex Anal. 2 (1995), 45–55.

[9] ———, *A compact set without Markov's property but with an extension operator for C^∞-functions,* Studia Math. 119 (1996), 27–35.

[10] A. Goncharov, T. Terzioğlu, and V. Zahariuta, *On isomorphic classification of tensor products $E_\infty(a)\hat{\otimes}E'_\infty(b)$,* Dissertationes Math., 350 (1996).

[11] A. P. Goncharov and V. P. Zahariuta, *Linear topological invariants and spaces of infinitely differentiable functions,* Math. Analiz i ego priloz., pp. 18–27, Rostov Univ., 1985 (Russian).

[12] ———, *On the existence of basis in spaces of Whitney functions on special compact sets in $\mathbb{R}$,* METU Preprint Series 93/58, Ankara, Turkey, 1993.

[13] M. Kocatepe and V. P. Zahariuta, *Köthe spaces modeled on spaces of C^∞-functions,* Studia Math. 121 (1996), 1–14.

[14] A. N. Kolmogorov, *On linear dimensionality of topological vector spaces,* Dokl. Akad. Nauk USSR 120 (1958), 239–341 (Russian).

[15] B. Malgrange, *Ideals of differentiable functions,* Oxford Univ. Press, London, 1967.

[16] B. S. Mitiagin, *Approximative dimension and bases in nuclear spaces,* Russian Math. Surveys 16, no. 4 (1961), 59–127.

[17] ———, *Sur l'equivalence des bases inconditionnelles dans les echelles de Hilbert,* C. R. Acad. Sci. Paris Sér. A-B 269 (1969), A426–A428.

[18] ———, *The equivalence of bases in Hilbert scales,* Studia Math. 37 (1970), 111–137 (Russian).

[19] A. Pełczyński, *On the approximation of S-spaces by finite dimensional spaces,* Bull. Acad. Pol. Sci. Cl. III. 5 (1957), 879–881.

[20] M. Tidten, *Fortsetzungen von C^∞-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}^n$ definiert sind,* Manuscripta Math. 27 (1979), 291–312.

[21] ———, *Kriterien für die Existenz von Ausdehnungsoperatoren zu $\mathcal{E}(K)$ für kompakte Teilmengen K von $\mathbb{R}$,* Arch. Math. (Basel) 40 (1983), 73–81.

[22] ———, *An example of a continuum of pairwise non-isomorphic spaces of C^∞ functions,* Studia Math. 78 (1984), 267–274.

[23] D. Vogt, *Characterisierung der Unterräume von s,* Math. Z. 155 (1977), 109–117.

[24] ———, *Charakterisierung der Unterräume eines nuklearen stabilen Potenzreihenraumes von endlichem Typ,* Studia Math. 71 (1982), 251–270.

[25] ———, *Sequence space representations of spaces of test functions and distributions,* Functional analysis, holomorphy and approximation theory (G. I. Zapata, ed.), pp. 405–443, Lecture Notes in Pure and Appl. Math., 83, Dekker, New York, 1983.

[26] ———, *Some results on continuous linear maps between Fréchet spaces,* Functional analysis: surveys and recent results III (K. D. Bierstedt, B. Fuchssteiner, eds.), pp. 349–381, North-Holland Math. Stud., 90, North-Holland, New York, 1984.

[27] D. Vogt and M. J. Wagner, *Charakterisierung der Quotienträume von s und eine Vermutung von Martineau,* Studia Math. 67 (1980), 225–240.

[28] M. J. Wagner, *Quotientenräume von stabilen Potenzreihenräumen endlichen Typs,* Manuscripta Math. 31 (1980), 97–109.

[29] M. Yurdakul and V. P. Zahariuta, *Linear topological invariants and isomorphic classification of Cartesian products of locally convex spaces,* Turkish J. Math. 19 (1995), 37–47.

[30] V. P. Zahariuta, *Linear topological invariants and the isomorphism of spaces of analytic functions,* Matem. analiz i ego pril., Rostov-on-Don, Rostov Univ., 2 (1970), 3–13; 3 (1971), 176–180 (Russian).
[31] ———, *Some linear topological invariants and isomorphisms of tensor products of centers of scales,* Izv. Sev. Kavkaz. Nauch. Centra Vysh. Skoly. 4 (1974), 62–64 (Russian).
[32] ———, *Generalized Mitiagin invariants and a continuum of pairwise nonisomorphic spaces of analytic functions,* Funktional Anal. i Prilozhen. 11 (1977), 24–30 (Russian).
[33] ———, *Synthetic diameters and linear topological invariants,* School on theory of operators in functional spaces (abstracts of reports), pp. 51–52, Minsk, 1978 (Russian).
[34] ———, *Linear topological invariants and their applications to isomorphic classification of generalized power spaces,* Rostov Univ., 1979 (Russian); revised English version in Turkish J. Math. 20 (1996), 237–289.
[35] ———, *Isomorphism of spaces of analytic functions,* Soviet Math. Dokl. 22 (1980), 631–634.
[36] ———, *On isomorphic classification of F-spaces,* Lecture Notes in Math., 1043, pp. 34–37, Springer, Berlin, 1984.

A. P. Goncharov
Department of Mathematics
Faculty of Science
Bilkent University
06533 Bilkent, Ankara
Turkey

and

Department of Mathematics
Rostov Building University
Rostov-na-Donu
Russia

goncha@fen.bilkent.edu.tr

M. Kocatepe
Department of Mathematics
Faculty of Science
Bilkent University
06533 Bilkent, Ankara
Turkey

kocatepe@fen.bilkent.edu.tr